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International Journal of Thermal Sciences 46 (2007) 118–127
International Journal of Thermal Sciences
www.elsevier.com/locate/ijts

Application of the heat-balance integral to an inverse Stefan problem ☆

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Received 23 February 2005; received in revised form 28 April 2006; accepted 28 April 2006

Available online 11 July 2006

Abstract

Most phase change process controls are concerned with the inverse Stefan problem. In this paper, the heat-balance integral method is applied effectively to analyze the one-region and two-region inverse Stefan problems in Cartesian and spherical coordinates. It is shown that if the movement of the phase change boundary is specified arbitrarily the present technique to predict both the temperature and its gradient at the fixed boundary is simple and accurate. As numerical illustrations, the one-dimensional inward solidification problem in Cartesian and spherical coordinates are solved and discussed in detail when the movement of the phase change interface is specified as a power function. The accuracy of these approximate solutions, based on the heat-balance integral method, is demonstrated satisfyingly by comparison with the available exact and/or numerical solutions for the one-region and the two-region problems.

Keywords: Inverse Stefan problem; Heat-balance integral method; Heat conduction; Phase change; Inward solidification problem; Cryopreservation; Cryosurgery

1. Introduction

In the Stefan problem the thermal condition, temperature and its gradient are usually specified at a fixed boundary and the conditions at the moving boundary, the history of the phase change boundary (a space–time curve), are to be solved by the governing equation together with the fixed boundary condition. In contrast to the direct Stefan problem, in the so-called inverse Stefan problem the thermal conditions are specified at the moving interface rather than the fixed boundary. In other words, the interface trajectory and its moving rate are known *a prior*. Both the temperature and its gradient required at the fixed boundary are to be determined so that the phase change boundary is moving at the specified rate. In brief, the phase change problem where the movement of the phase change boundary is specified rather than to be solved for, is known as an inverse Stefan problem.

It should be noted that the solution of the inverse Stefan problem not only provides information on the behavior of the Stefan problem, but also has wide practical applications in physics and technique: e.g., crystallization of metal in metallurgy, controlled fluidized bed baths for surface coating and controlled ablation of thermal shields in aerospace engineering, etc. It is also applied to the cryopreservation of cells, and cryosurgery [1,2]. One of the most important factors for cells destruction or survival is the cooling rate during cryopreservation of biomaterials by freezing or vitrification. Studies in thermodynamics and crystallization kinetics could provide available information to estimate the cooling rates of cells solidified into an ice crystal state or a noncrystalline glassy state [3]. On the other hand, the determination and the control of the cooling rate at the interface of phase change require solving the heat conduction problem with phase change, direct and/or inverse Stefan problem.

Langford [4] obtained the solutions of the one-dimensional inverse Stefan problem by a series expansion. When the moving rate of the phase change interface is constant the series solution reduced to the classical solution given by Stefan. Rubinsky and Shitzer [1] obtained a long time analytic solution by means of the integration of the governing equation and analytic iterations. The solutions of the above methods are expressed in infinite series. Chow and Woo [5] suggested a numerical method based on the Laplace transform. The inverse Laplace transform is per-

☆ The research reported in this paper has supported by National Nature Science Foundation of China.
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1290-0729/$ – see front matter
doi:10.1016/j.ijthermalsci.2006.04.013

Nomenclature

c_p	specific heat
i	$i = 0, 1$ for plane, sphere, respectively
L	latent heat
n	index of power function in Eq. (15)
r	spatial coordinate
r_0	position of the fixed boundary
R	dimensionless spatial coordinate
s	position of phase change interface
S	dimensionless position of phase change interface
t	time
T	temperature
T_0	reference temperature in Eq. (2) or initial temperature in Eq. (34)
T_{ph}	phase change temperature
u	dimensionless temperature defined in Eqs. (2) and (34)
Ste	Stefan number defined in Eq. (2)

Greek symbols

α	thermal diffusivity
β	constant in Eq. (15)
γ	Euler's constant
δ	thermal layer thickness
ρ	density
λ	thermal conductivity
τ	dimensionless time defined in Eq. (2)
θ	dimensionless temperature defined in Eqs. (3) and (34)
Θ	defined in Eqs. (6) and (37)

Subscripts

l	liquid phase
s	solid phase

formed by using a statistical technique. Jiang et al. [6] chose the Stefan number, *Ste*, as a small parameter and obtained the perturbation solution for a planar inverse Stefan problem. The heat-balance integral was suggested to solve the one-dimensional inverse Stefan problem [2], which can be applied to cryopreservation of biomaterials. Using an integral method, Zabaras et al. [7] provided the solution of a two-phase inverse heat transfer problem with phase change. Due to given temperature gradients of both liquid and solid phases on the moving interface, the solution is equivalent to the solution of two separate inverse moving boundary problems. Budman et al. [8] presented an integral solution for inverse Stefan problem of a nonideal binary solution. The condition of a constant cooling/thawing rate at the freezing front was assumed. A combined solution of the one-dimensional and semi-infinite inverse Stefan problem in biological tissues was presented by Rabin and Shitzer [9]. The analysis combines a heat balance integral solution in the frozen region and a numerical enthalpy-based solution approach in the unfrozen region.

The heat-balance integral method was introduced previously to analyze the direct phase change problem by Goodman [10], and applied to the three-dimensional problem of the freezing of a cuboid by Riley and Duck [11]. Bell [12] suggested a refinement of the heat balance integral method with equal subdivision of the dependent variable temperature. More information has been given in the review book by Crank [13]. Caldwell and Chiu [14] extended this method to the two-phase Stefan problem in cylindrical and spherical geometries. Mosally et al. investigated the use of several exponential functions both for whole-domain solutions and for piecewise solutions [15].

The purpose of the present study is using the heat-balance integral method to solve the one-dimensional inward solidification inverse Stefan problem, when the movement of the phase change interface is specified. The one-region problem with the freezing of a liquid layer initially at the phase change temperature is analyzed firstly by using a cubic polynomial temperature profile for solid phase. The more general two-region problem is then considered, in which the initial liquid temperature is higher than the freezing temperature. The cubic polynomial temperature distributions are chosen for both the liquid and solid phases. As numerical illustrations, the approximate analytic solutions expressed in finite forms for both the Cartesian and spherical geometries are obtained when the movement of the phase change interface is specified as $S = 1 - (\beta\tau)^{1/n}$. The results are in good agreement compared with the exact or numerical solutions given in the literature.

2. Analysis

2.1. One-region problem

2.1.1. Governing equations

The problem of the one-dimensional inward solidification or freezing is considered here. It is assumed that the entire domain is initially the liquid at the phase change temperature. Along with solidification or freezing at the fixed boundary at time zero, the domain is separated into the solid and liquid phase by the phase change interface. The liquid phase being at a constant phase change temperature T_{ph} throughout, the temperature is unknown only in the solid phase, so the problem is a one-region problem. The location of the moving interface is a monotonic function of time. As for the inverse Stefan problem, the variation of the interface with time is specified and controlled, i.e., it is a known function of time. At this moving boundary the temperature of the solid is maintained at the phase change temperature and the latent heat released by the liquid during the solidification is equal to the heat conducted into the solid. The characteristic is shown in Fig. 1.

Since the constant thermophysical properties are assumed, the problem can be formulated by the following governing equation of the solid and the boundary conditions at the phase change interface:

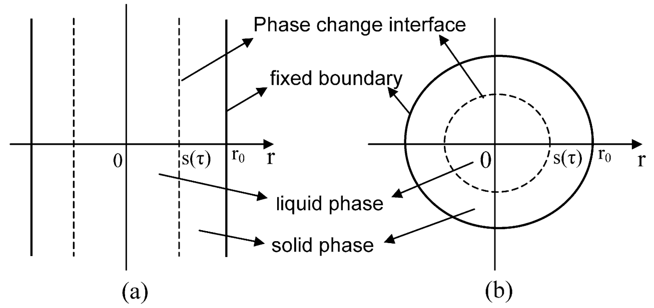

Fig. 1. Schematic drawing of model for (a) Cartesian geometry, (b) spherical geometry.

$$\frac{1}{r^i}\frac{\partial^2}{\partial r^2}\left(r^i T_s\right)=\frac{1}{\alpha_s}\frac{\partial T_s}{\partial t},\quad s(t)<r\leqslant r_0,\quad t>0 \tag{1a}$$

$$T_s=T_{\mathrm{ph}},\quad r=s(t),\quad t>0 \tag{1b}$$

$$\lambda_s\frac{\partial T_s}{\partial r}=\rho L\frac{\mathrm{d}s}{\mathrm{d}t},\quad r=s(t),\quad t>0 \tag{1c}$$

$$s(0)=r_0 \tag{1d}$$

where r_0 is the position of the fixed boundary.

It should be noted that the governing equation (1a) can be applicable for either Cartesian or spherical geometry, i.e., $i=0$ or 1 for plane and sphere, respectively. The mathematical formulations are also valid for outward solidification problem as well as inward or outward melting problem. The latent heat L in Eq. (1c) is taken positive for solidification and negative for melting. The moving rate of the phase change interface, $\mathrm{d}s/\mathrm{d}t$, is taken negative value for inward problem and positive value for outward problem.

Now define the following nondimensional parameters:

$$u_s=\frac{T_s-T_{\mathrm{ph}}}{T_0-T_{\mathrm{ph}}},\qquad R=\frac{r}{r_0},\qquad \tau=\frac{\alpha_s t}{r_0^2}$$

$$S=\frac{s}{r_0},\qquad Ste=\frac{c_p(T_0-T_{\mathrm{ph}})}{L} \tag{2}$$

where T_0 is an external reference temperature. A new variable $\theta(R,\tau)$ is defined as:

$$\theta_s=R^i u_s \tag{3}$$

Then, Eqs. (1a)–(1d) can be nondimensionalized as follows:

$$\frac{\partial^2\theta_s}{\partial R^2}=\frac{\partial\theta_s}{\partial\tau},\quad S(\tau)<R\leqslant 1,\quad \tau>0 \tag{4a}$$

$$\theta_s=0,\quad R=S(\tau),\quad \tau>0 \tag{4b}$$

$$\frac{\partial\theta_s}{\partial R}=\frac{S^i}{Ste}\frac{\mathrm{d}S}{\mathrm{d}\tau},\quad R=S(\tau),\quad \tau>0 \tag{4c}$$

$$S(0)=1 \tag{4d}$$

2.1.2. Heat-balance integral method

Goodman [10] used the heat-balance integral method to solve a one-dimensional transient melting problem. The basic steps of this method can be summarized as follows. First define a thermal layer thickness, which is identical to the definition of the location of the phase change interface, $S(\tau)$. A suitable profile, generally a polynomial profile, is then chosen for the temperature distribution over the thermal layer. The coefficients in the polynomial are determined by utilizing the fixed and moving boundary conditions in terms of the thermal layer thickness, i.e., the position of the interface. After the temperature profile is introduced into the heat-balance integral equation, an ordinary differential equation is obtained for the location of the phase change interface. The solution of this differential equation gives the moving interface position as a function of time. Substituting $S(\tau)$ into the polynomial profile, the temperature distribution is finally determined completely.

When the heat-balance integral method is applied to the inverse Stefan problem, it is not necessary to determine the functional relation between the position of the moving boundary and the time, because it has been specified *a priori*. Therefore, a polynomial temperature profile which satisfies all the moving boundary conditions and consists of a number of adjustable coefficients can be directly inserted into the heat-balance integral equation. We now summarize the basic steps in the solution of the inverse Stefan problem with the heat-balance integral method. At first the governing equation is integrated with respect to the space variable from the fixed boundary to the moving boundary. As a result, an integral equation is obtained. A selected polynomial temperature profile, wherein some of the coefficients are determined by the moving boundary conditions, is then substituted into the heat-balance integral equation. The resulting equation is an ordinary differential equation for the undetermined coefficient. Solving this ordinary differential equation, the undetermined coefficient and the temperature distribution satisfied the moving boundary conditions and the heat-balance integral equation can be obtained finally.

The heat-balance integral method stated above is now applied to the one-dimensional inward inverse Stefan problem. The integration of governing equation (4a) from $R=1$ to $R=S(\tau)$ gives following heat-balance integral equation

$$\left(\frac{\partial\theta_s}{\partial R}\right)_{R=S}-\left(\frac{\partial\theta_s}{\partial R}\right)_{R=1}=\frac{\mathrm{d}\Theta_s}{\mathrm{d}\tau} \tag{5}$$

The Leibnitz's rule and condition (4b) have been used and defined

$$\Theta_s=\int_1^S\theta_s\,\mathrm{d}R \tag{6}$$

To solve this equation we assume a cubic polynomial representation for θ_s in the form

$$\theta_s=a+b\left(\frac{R-S}{1-S}\right)+c\left(\frac{R-S}{1-S}\right)^2+d\left(\frac{R-S}{1-S}\right)^3 \tag{7}$$

where the coefficients are in general functions of $S(\tau)$. Four conditions are required to determine these coefficients. Eqs. (4b) and (4c) provide two conditions to determine a and b. Another condition can be developed as described by Goodman [10,13].

Eq. (4b) is differentiated with respect to τ

$$\frac{\partial\theta_s}{\partial R}\frac{\mathrm{d}S}{\mathrm{d}\tau}+\frac{\partial\theta_s}{\partial\tau}=0 \tag{8}$$

Substituting from Eq. (4c) for $\partial\theta_s/\partial R$ and Eq. (4a) for $\partial\theta_s/\partial\tau$, an additional derived condition at moving interface is then obtained:

$$\frac{\partial^2\theta_s}{\partial R^2} = -\frac{S^i}{Ste}\left(\frac{\mathrm{d}S}{\mathrm{d}\tau}\right)^2 \tag{9}$$

Three of these coefficients can be determined by three conditions (4b), (4c) and (9):

$$a = 0, \qquad b = \frac{(1-S)S^i}{Ste}\left(\frac{\mathrm{d}S}{\mathrm{d}\tau}\right)$$

$$c = -\frac{(1-S)^2 S^i}{2\cdot Ste}\left(\frac{\mathrm{d}S}{\mathrm{d}\tau}\right)^2 \tag{10}$$

From Eqs. (6) and (7), a relation between the undetermined coefficient d and other coefficients and Θ_s can then be obtained as follows:

$$d = -4\left(\frac{\Theta_s}{1-S} + \frac{b}{2} + \frac{c}{3}\right) \tag{11}$$

Introducing Eq. (7) together with Eqs. (10) and (11) into Eq. (5) and performing the indicated operations, we obtain the following first-order linear ordinary differential equation for the determination of Θ_s

$$\frac{\mathrm{d}\Theta_s}{\mathrm{d}S} = \frac{12\Theta_s}{(1-S)^2}\left(\frac{\mathrm{d}S}{\mathrm{d}\tau}\right)^{-1} + \frac{6}{Ste}S^i - \frac{(1-S)S^i}{Ste}\left(\frac{\mathrm{d}S}{\mathrm{d}\tau}\right) \tag{12}$$

When the variation of the phase change interface with time is given Θ_s can be easily solved. The coefficient d is then determined from Eq. (11). Finally, all of four coefficients in the selection temperature profile are completely determined.

For the control-type inverse Stefan problem, we do not pay attention to the temperature distribution in the entire domain. We are interested in what are the thermal conditions required at the fixed boundary so that the phase change boundary is moving at the specified rate. In other words, both the temperature and its gradient, namely, the heat flow rate at the fixed boundary, should be solved. From Eqs. (3), (7), (9) and (11) as well as Θ_s solved above, the temperature and its gradient at the fixed boundary are respectively given as

$$u_s(1,\tau) = -\frac{4\Theta_s}{1-S} - \frac{(1-S)S^i}{Ste}\left(\frac{\mathrm{d}S}{\mathrm{d}\tau}\right) + \frac{(1-S)^2 S^i}{6\cdot Ste}\left(\frac{\mathrm{d}S}{\mathrm{d}\tau}\right)^2 \tag{13}$$

$$\frac{\partial u_s(1,\tau)}{\partial R} = -\frac{4(2+S^i)}{(1-S)^2}\Theta_s - \frac{(4+S^i)S^i}{Ste}\left(\frac{\mathrm{d}S}{\mathrm{d}\tau}\right) + \frac{(5+S^i)S^i(1-S)}{6\cdot Ste}\left(\frac{\mathrm{d}S}{\mathrm{d}\tau}\right)^2 \tag{14}$$

where $i = 0$ for Cartesian geometry; $i = 1$ for spherical geometry.

2.1.3. Numerical illustrations

In this section, some examples are given, to illustrate the application of the heat-balance integral method in the inverse Stefan problem. We consider the one-dimensional inward solidification problem subject to different conditions prescribed at the moving boundary. The solutions are presented for both the Cartesian and spherical geometries and compared with the exact or numerical solutions.

Cartesian geometry ($i = 0$). This corresponds to a planar inverse Stefan problem (Fig. 1(a)). We suppose that the phase change interface varies with time following the power law:

$$S(\tau) = 1 - (\beta\tau)^{1/n} \tag{15}$$

Hence, its moving rate is

$$\frac{\mathrm{d}S}{\mathrm{d}\tau} = -\beta\frac{(1-S)^{1-n}}{n} \tag{16}$$

where both n and β are positive constants. Substituting the above equation into Eq. (12), we obtain the following ordinary differential equation for the determination of Θ_s

$$\frac{\mathrm{d}\Theta_s}{\mathrm{d}S} = A(S)\Theta_s + B(S) \tag{17}$$

where

$$A(S) = -\frac{12n}{(1-S)^{3-n}\beta}, \qquad B(S) = \frac{6}{Ste} + \frac{\beta(1-S)^{2-n}}{n\cdot Ste} \tag{18}$$

We note that, the nonhomogeneous term $B(S)$ on the right-hand side of Eq. (18) is a constant when $n = 2$, and is a positive power function of $(1-S)$ when $n < 2$. Thus, for $n \leqslant 2$ the condition needed to determine the integral constant in the general solution can be given as $\Theta_s(1) = 0$, by Eq. (8). However, $B(S)$ is a negative power function of $(1-S)$ when $n > 2$. It is meaningless at $S = 1$. We also note that before the phase change boundary reaches the end of the slab, the problem is the same as that of a semi-infinite slab in which the temperature of the liquid phase is maintained at phase change temperature throughout [5]. Hence, as $R \to \infty$, thus $S(\tau) \to \infty$, the temperature distribution should be a finite value. Therefore, for the case $n > 2$, the condition needed to determine the integral constant is given as $\Theta_s(S) \to$ finite.

Knowing Θ_s, the temperature and its gradient at the fixed boundary are then determined from Eqs. (13) and (14), respectively. To compare with the exact or numerical solutions, the temperature and its gradient at the fixed boundary are given for these different values of the parameter n, namely, 1,2 and 3 respectively.

(1) $n = 1$, $S(\tau) = 1 - \beta\tau$

$$u_s(1,\tau) = \frac{1}{Ste}\left\{3\beta^2\tau + \frac{\beta^4\tau^2}{6} + 48\left[1 - \frac{12}{\beta^2\tau}e^{-12/(\beta^2\tau)}\left(Ei\left(\frac{12}{\beta^2\tau}\right) - \gamma\right)\right]\right\} \tag{19}$$

$$\frac{\partial u_s(1,\tau)}{\partial R} = \frac{1}{Ste}\left\{11\beta + \beta^2\tau + \frac{144}{\beta\tau}\left[1 - \frac{12}{\beta^2\tau}e^{-12/(\beta^2\tau)}\left(Ei\left(\frac{12}{\beta^2\tau}\right) - \gamma\right)\right]\right\} \tag{20}$$

Euler's constant $\gamma = 0.57721566$ and $Ei(x)$ is an exponential integral function [16]. As the phase change interface is moving at a constant rate, the exact solution is known as classical Stefan's solution [17]:

$$u_s(1,\tau) = \frac{1}{Ste}\left(1 - e^{\beta^2\tau}\right) \tag{21}$$

$$\frac{\partial u_s(1,\tau)}{\partial R} = -\frac{\beta}{Ste}e^{\beta^2\tau} \tag{22}$$

(2) $n = 2,\ S(\tau) = 1 - (\beta\tau)^{1/2}$

$$u_s(1,\tau) = \frac{\beta(\beta^2 + 36\beta + 288)}{24 \cdot Ste(\beta - 24)} \tag{23}$$

$$\frac{\partial u_s(1,\tau)}{\partial R} = \frac{\sqrt{\beta}(\beta^2 + 10\beta + 48)}{4\sqrt{\tau} \cdot Ste(\beta - 24)} \tag{24}$$

For this case, the exact solution is given by Neumann as [17]:

$$u_s(1,\tau) = -\frac{\sqrt{\pi\beta}}{2 \cdot Ste}e^{\beta/4}erf\left(\frac{\sqrt{\beta}}{2}\right) \tag{25}$$

$$\frac{\partial u_s(1,\tau)}{\partial R} = -\frac{\sqrt{\beta}}{2 \cdot Ste}e^{\beta/4}\tau^{-1/2} \tag{26}$$

(3) $n = 3,\ S(\tau) = 1 - (\beta\tau)^{1/3}$

$$u_s(1,\tau) = \frac{\beta}{Ste}\left\{-\frac{(\beta\tau)^{-1/3}}{3} + \frac{\beta^{1/3}\tau^{-2/3}}{27}\left[\frac{1}{2} - 36\beta^{-2/3} \times \tau^{1/3}e^{36\beta^{-2/3}\tau^{1/3}}E_1\left(36\beta^{-2/3}\tau^{1/3}\right)\right]\right\} \tag{27}$$

$$\frac{\partial u_s(1,\tau)}{\partial R} = \frac{\beta}{Ste}\left\{-\frac{(\beta\tau)^{-2/3}}{3} + \frac{\tau^{-1}}{9}\left[1 - 36\beta^{-2/3}\tau^{1/3} \times e^{36\beta^{-2/3}\tau^{1/3}}E_1\left(36\beta^{-2/3}\tau^{1/3}\right)\right]\right\} \tag{28}$$

where $E_1(x) = -Ei(-x)$. There is no known exact solution to this case. Using a Laplace transform technique Chow and Woo [5] obtained the numerical results for the case $n = 3$, $Ste = 1$ and $\beta = 1$. The same numerical results were also given by Frederick and Grief [18].

In Tables 1–3, the numerical values of the temperature and its gradient at the fixed boundary in Cartesian geometry are given for $Ste = 1$, $\beta = 1$, $n = 1, 2, 3$, respectively. For these cases, the present method based on the heat-balance integral yields results that agree reasonably well with the exact or numerical solution for $0.1 \leqslant \tau \leqslant 1$. When $n = 1$, the present results seem to agree with the exact solution better than that of Chow and Woo [5]. At $\tau = 0.8$, the relative error of the result given in [5] is 24%, whereas the one based on the heat-balance integral method is only 2% at $\tau = 1$. When $n = 2$ and 3, the maximum error is less than 0.6% and 1%, respectively. At $\tau = 1$, the position of the phase change interface $S = 0$, the entire domain has been solidified so that the problem is transformed to the pure solid phase conduction when $\tau > 1$. From this time on, the calculated results have no longer the physical significance of the phase-change heat conduction.

The parameter β has an effect on the solution. With the increase of β the relative error between the exact and the heat-balance integral approximate solution will be enlarged. With decreasing β, it will be reduced. For example, when $\beta = 0.5$, $n = 1$ and 2, the maximum error is decreased to 0.18% and 0.15% respectively. If β decreases still further, the maximum error will become smaller. So we can say, when $\beta \leqslant 1$ the maximum error between the present and exact solution is not greater than 2%.

It is noteworthy that the effect of the Stefan number, *Ste*, on solutions obtained by the heat-balance integral method is consistent with exact solutions. Therefore, unlike the perturbation solution, which is applied only to small *Ste*, the heat-balance integral method can be applied whether the Stefan number is large or small.

It can be seen from the above solutions and analyses of Eq. (18) that there is a critical case concerning the temperature of the fixed boundary when the phase change interface varies with time following a power law. The temperature at the fixed boundary, $u(1,\tau)$, is a constant independent of time when $n = 2$. When $n < 2$, $u(1,\tau)$ is equal to zero at $\tau = 0$, and then decreases with time. While $n > 2$, $u(1,\tau)$ approaches $-\infty$ at $\tau = 0$, then increases with time so as to approach the

Table 1
The temperature and its gradient at the fixed boundary in Cartesian geometry for $Ste = 1$, $S = 1 - \tau$

τ	$u_s(1,R)$			$\partial u_s(1,R)/\partial R$		
	Stefan [17]	Chow [5]	present	Stefan [17]	Chow [5]	present
0.1	−0.1052	−0.1052	−0.1052	−1.1052	−1.1048	−1.1050
0.3	−0.3499	−0.3510	−0.3499	−1.3499	−1.3510	−1.3495
0.5	−0.6487	−0.6483	−0.6499	−1.6487	−1.6306	−1.6493
0.7	−1.0138	−0.8701	−1.0198	−2.0138	−1.8335	−2.0206
0.9	−1.4596		−1.4811	−2.4596		−2.4869
1.0	−1.7183		−1.7552	−2.7183		−2.7656

Table 2
The temperature and its gradient at the fixed boundary in Cartesian geometry for $Ste = 1$, $S = 1 - \tau^{1/2}$

τ	$u_s(1,R)$			$\partial u_s(1,R)/\partial R$		
	Neumann [17]	Chow [5]	present	Neumann [17]	Chow [5]	present
0.1	−0.5923	−0.5923	−0.5888	−2.0302	−2.0304	−2.0280
0.3	−0.5923	−0.5923	−0.5888	−1.1721	−1.1725	−1.1709
0.5	−0.5923	−0.5923	−0.5888	−0.9079	−0.9080	−0.9069
0.7	−0.5923	−0.5923	−0.5888	−0.7674	−0.7675	−0.7665
0.9	−0.5923		−0.5888	−0.6767		−0.6760
1.0	−0.5923		−0.5888	−0.6420		−0.6413

Table 3
The temperature and its gradient at the fixed boundary in Cartesian geometry for $Ste = 1$, $S = 1 - \tau^{1/3}$

τ	$u_s(1,R)$			$\partial u_s(1,R)/\partial R$		
	Frederick [18]	Chow [5]	present	Frederick [18]	Chow [5]	present
0.1	−0.8035	−0.8036	−0.7949	−1.4744	−1.4759	−1.4875
0.3	−0.5391	−0.5392	−0.5361	−0.7274	−0.7275	−0.7290
0.5	−0.4493	−0.4494	−0.4474	−0.5209	−0.5210	−0.5219
0.7	−0.3989	−0.3989	−0.3975	−0.4176	−0.4177	−0.4181
0.9	−0.3651		−0.3644	−0.3539		−0.3542
1.0	−0.3518		−0.3509	−0.3301		−0.3304

phase change temperature. These conclusions are consistent with Ref. [6].

Spherical geometry ($i = 1$). As an illustration to the results in spherical coordinates the simple case of a constant rate of the moving boundary is studied. The movement of the phase change interface as a function of time is assumed to be $S = 1 - \beta\tau$. Once Θ_s is determined from the solution of the ordinary differential equation (12) together with the condition $\Theta_s(1) = 0$, the temperature and its gradient at the fixed boundary can be solve from Eqs. (13) and (14), respectively. We obtain

$$u_s(1,\tau) = \frac{1}{Ste}\left\{-\frac{1}{6}\beta^5\tau^3 + \frac{1}{6}\beta^3(\beta-14)\tau^2 + \beta(3\beta-20)\tau - \frac{48(5-\beta)}{\beta}\left[1 - \frac{12}{\beta^2\tau}e^{-12/(\beta^2\tau)} \times \left(Ei\left(\frac{12}{\beta^2\tau}\right) - \gamma\right)\right]\right\} \tag{29}$$

$$\frac{\partial u_s(1,\tau)}{\partial R} = \frac{1}{Ste}\left\{\frac{1}{6}\beta^5\tau^3 + \frac{7}{6}\beta^3(\beta-2)\tau^2 + \beta(\beta-2)(\beta-10)\tau + 11\beta - 60 - \frac{48(3-\beta\tau)(5-\beta)}{\beta^2\tau} \times \left[1 - \frac{12}{\beta^2\tau}e^{-12/(\beta^2\tau)}\left(Ei\left(\frac{12}{\beta^2\tau}\right) - \gamma\right)\right]\right\} \tag{30}$$

The exact solution of the same problem is obtained by Langford [4] as

$$u_s(1,\tau) = -\frac{1}{Ste}\left\{\left(1+\frac{2}{\beta}\right)\left(e^{\beta^2\tau} - 1\right) - 2\beta\tau e^{\beta^2\tau}\right\} \tag{31}$$

$$\frac{\partial u_s(1,\tau)}{\partial R} = -\frac{1}{Ste}\left\{\frac{2}{\beta} + \left(\beta - \frac{2}{\beta} + 2\beta\tau - 2\beta^2\tau\right)e^{\beta^2\tau}\right\} \tag{32}$$

A comparison of the approximate and exact solution is given in Fig. 2. The agreement is good for $0 \leqslant \tau \leqslant 1$. It can be seen clearly from Fig. 2 that the one-dimensional inward solidification problem in spherical geometry is different from that in Cartesian geometry. The temperature gradient at the fixed boundary, $\partial u(1, R)/\partial R$, namely the heat flux, is negative when $\tau < 0.69$. This is meant that heat is removed from the fixed boundary. While $\tau > 0.69$, the temperature gradient is positive if the required phase change interface continues at the same constant velocity to the center of the sphere, i.e., it is then necessary to add rather than to remove heat from the fixed boundary.

As for the temperature at the fixed boundary, $u(1,\tau)$, it is initially equal to the phase change temperature. Then, $u(1,\tau)$ will decrease gradually with time until the phase change interface has moved halfway to the center of the sphere. At $\tau = 0.5$, $u(1,\tau)$ decreases to the lowest value. If a constant velocity is required to continue the movement of the phase change interface, it is necessary that $u(1,\tau)$ increase with time. As a result, $u(1,\tau) = 0$ (T_s returns to the phase change temperature) at about $\tau = 0.87$. With the movement of the interface to the center of the sphere $u(1,\tau)$ will increase with time as a positive value, namely $u(1,\tau) > 0$. This means that the solidified domain adjoined to the fixed boundary will be melted again and becomes warmer in order to keep the phase change interface continuously moving at the constant velocity. However, it is contradictory to the problem of inward solidification. Therefore the solution cannot be valid for $\tau > 0.87$ and the problem has to be reformulated.

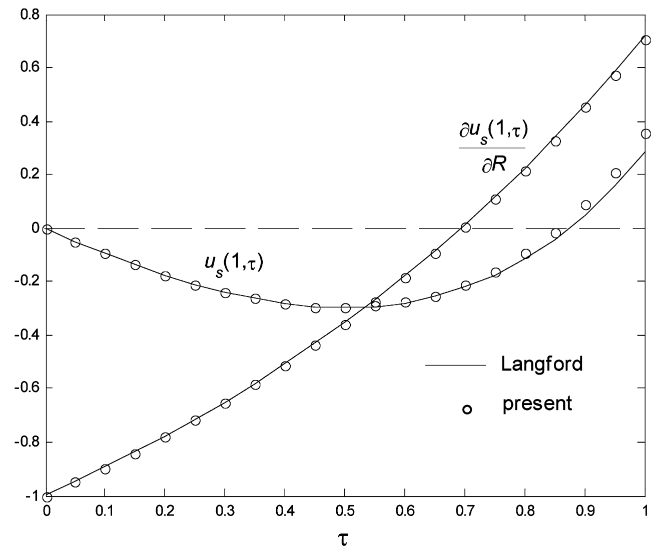

Fig. 2. The temperature and its gradient at the fixed boundary for the inward solidification problem in spherical geometry ($Ste = 1$, $S = 1 - \tau$).

2.2. Two-region problem

2.2.1. Governing equations

When the initial liquid temperature T_0 is higher than the freezing temperature T_{ph}, the inverse Stefan problem is a two-region problem because the temperatures are unknown in both the solid and liquid phases. In the following analysis the one-dimensional inward solidification with Cartesian and spherical coordinate systems confined in a finite medium, $0 \leqslant R \leqslant 1$ are considered. The governing equations with initial condition and moving boundary conditions are as follows in the dimensionless form:

For the liquid phase:

$$\frac{\partial^2\theta_l}{\partial R^2} = \frac{\alpha_s}{\alpha_l}\frac{\partial\theta_l}{\partial\tau}, \quad 0 \leqslant R < S(\tau), \quad \tau > 0 \tag{33a}$$

$$\theta_l = R^i, \quad \tau = 0 \tag{33b}$$

For the solid phase:

$$\frac{\partial^2\theta_s}{\partial R^2} = \frac{\partial\theta_s}{\partial\tau}, \quad S(\tau) < R \leqslant 1, \quad \tau > 0 \tag{33c}$$

and for the solid–liquid interface:

$$\theta_l = \theta_s = 0, \quad R = S(\tau) \tag{33d}$$

$$\frac{\partial\theta_s}{\partial R} - \frac{\lambda_l}{\lambda_s}\frac{\partial\theta_l}{\partial R} = \frac{S^i}{Ste}\frac{dS}{d\tau}, \quad R = S(\tau) \tag{33e}$$

$$S(0) = 1 \tag{33f}$$

It is the same as the one-region problem, $i = 0, 1$ for Cartesian geometry, sphere geometry, respectively. Some dimensionless quantities are defined in (2) and (3), others are defined as

$$u_l = \frac{T_l - T_{\text{ph}}}{T_0 - T_{\text{ph}}}, \quad \theta_l = R^i u_l \tag{34}$$

where T_0 is the initial temperature for the two-region problem.

2.2.2. Heat-balance integral method

To apply the heat-balance integral method for the two-region inverse problem, besides the interface of phase change, another thermal layer $\delta(\tau)$ is defined as the penetration distance beyond which, for practical purposes, there is no heat flow (i.e., $\partial u_l/\partial R = 0$ at $R = \delta(\tau)$); hence the initial temperature distribution remains unaffected (i.e., $u_l(\delta(\tau), \tau) = 1$ at $R = \delta(\tau)$); An additional smooth condition (i.e., $\partial^2 u_l/\partial R^2 = 0$ at $R = \delta(\tau)$) can be derived as given by Goodman [10]. For the finite region considered here the thermal layer concept is valid so long as $0 < \delta(\tau) \leqslant 1$. The heat conduction equations (33a) for the liquid phase and (33c) for the solid phase are integrated from $R = S(\tau)$ to $R = \delta(\tau)$ and $R = 1$ to $R = S(\tau)$, respectively. Utilizing condition (33d), heat-balance integral equations for liquid phase and solid phase respectively are obtained as follows

$$\left(\frac{\partial \theta_l}{\partial R}\right)_{R=\delta} - \left(\frac{\partial \theta_l}{\partial R}\right)_{R=S} = \frac{\alpha_s}{\alpha_l}\left[\frac{\mathrm{d}\Theta_l}{\mathrm{d}\tau} - \theta_l(\delta, \tau)\frac{\mathrm{d}\delta}{\mathrm{d}\tau}\right] \tag{35}$$

$$\left(\frac{\partial \theta_s}{\partial R}\right)_{R=S} - \left(\frac{\partial \theta_s}{\partial R}\right)_{R=1} = \frac{\mathrm{d}\Theta_s}{\mathrm{d}\tau} \tag{36}$$

where Θ_s is defined by Eq. (6) and

$$\Theta_l = \int_S^\delta \theta_l \,\mathrm{d}R \tag{37}$$

A cubic polynomial representation for θ_l is chosen as

$$\theta_l = A_l + B_l\left(\frac{R-\delta}{S-\delta}\right) + C_l\left(\frac{R-\delta}{S-\delta}\right)^2 + D_l\left(\frac{R-\delta}{S-\delta}\right)^3 \tag{38a}$$

By the definition of the thermal layer described above and Eq. (34), we have three conditions at $R = \delta(\tau)$ needed to determine these four coefficients as

$$R = \delta(\tau), \qquad \theta_l = \delta^i, \qquad \frac{\partial \theta_l}{\partial R} = i, \qquad \frac{\partial^2 \theta_l}{\partial R^2} = 0 \tag{39}$$

Introducing Eqs. (39) and (33d) into (38a) we obtain

$$\theta_l = R^i - S^i\left(\frac{R-\delta}{S-\delta}\right)^3 \tag{38b}$$

For solid phase θ_s, a cubic temperature profile is also chosen, similarly to (7), as

$$\theta_s = A_s + B_s\left(\frac{R-S}{1-S}\right) + C_s\left(\frac{R-S}{1-S}\right)^2 + D_s\left(\frac{R-S}{1-S}\right)^3 \tag{40}$$

The coefficients A_s, B_s and C_s are determined by the application of the interface conditions (33d), (33e) and an additional derived condition

$$R = S(\tau), \quad \frac{\partial^2 \theta_s}{\partial R^2} - \frac{\lambda_l}{\lambda_s}\frac{\alpha_l}{\alpha_s}\frac{\partial^2 \theta_l}{\partial R^2} = -\frac{S^i}{Ste}\left(\frac{\mathrm{d}S}{\mathrm{d}\tau}\right)^2 \tag{41}$$

which can be obtained by differentiating equation (33d) with respect to time and applying Eqs. (33a), (33c) and (33e), as in the one-region problem discussed above. We obtain

$$A_s = 0 \tag{42a}$$

$$B_s = (1-S)\left[\frac{S^i}{Ste}\left(\frac{\mathrm{d}S}{\mathrm{d}\tau}\right) + \frac{\lambda_l}{\lambda_s}\left(i - \frac{3S^i}{S-\delta}\right)\right] \tag{42b}$$

$$C_s = \frac{(1-S)^2}{2}\left[-\frac{S^i}{Ste}\left(\frac{\mathrm{d}S}{\mathrm{d}\tau}\right)^2 - \frac{\lambda_l}{\lambda_s}\frac{\alpha_l}{\alpha_s}\frac{6S^i}{(S-\delta)^2}\right] \tag{42c}$$

From Eqs. (37) and (40), a relation between the coefficient D_s and other coefficients and Θ_s can then be obtained as follows:

$$D_s = -4\left(\frac{\Theta_s}{1-S} + \frac{B_s}{2} + \frac{C_s}{3}\right) \tag{42d}$$

Introducing Eqs. (40) and (42) into Eq. (36), the heat-balance integral equation for solid phase becomes

$$\begin{aligned}\frac{\mathrm{d}\Theta_s}{\mathrm{d}S} = {} & \frac{12\Theta_s}{(1-S)^2}\left(\frac{\mathrm{d}S}{\mathrm{d}\tau}\right)^{-1} \\ & + 6\left[\frac{S^i}{Ste} + \frac{\lambda_l}{\lambda_s}\left(i - \frac{3S^i}{S-\delta}\right)\left(\frac{\mathrm{d}S}{\mathrm{d}\tau}\right)^{-1}\right] \\ & - (1-S)\left[\frac{S^i}{Ste}\left(\frac{\mathrm{d}S}{\mathrm{d}\tau}\right) \right. \\ & \left. + \frac{\lambda_l}{\lambda_s}\frac{\alpha_l}{\alpha_s}\frac{6S^i}{(S-\delta)^2}\left(\frac{\mathrm{d}S}{\mathrm{d}\tau}\right)^{-1}\right]\end{aligned} \tag{43}$$

Similar to Eq. (12) for the one-region problem, Eq. (43) is also a first-order linear ordinary differential equation. Once Θ_s is determined from the solution of above equation, D_s is evaluated from Eq. (42d). Both the temperature and its gradient at the fixed boundary $R = 1$, can be solved from Eqs. (40), (42) and (3) as follows

$$\begin{aligned}u_s(1,\tau) = {} & -\frac{4\Theta_s}{1-S} - (1-S)\left[\frac{S^i}{Ste}\left(\frac{\mathrm{d}S}{\mathrm{d}\tau}\right) + \frac{\lambda_l}{\lambda_s}\left(i - \frac{3S^i}{S-\delta}\right)\right] \\ & + (1-S)^2\left[\frac{S^i}{6\cdot Ste}\left(\frac{\mathrm{d}S}{\mathrm{d}\tau}\right)^2 + \frac{\lambda_l}{\lambda_s}\frac{\alpha_l}{\alpha_s}\frac{S^i}{(S-\delta)^2}\right]\end{aligned} \tag{44}$$

$$\begin{aligned}\frac{\partial u_s(1,\tau)}{\partial R} = {} & -\frac{4(2+S^i)}{(1-S)^2}\Theta_s \\ & - \left(4+S^i\right)\left[\frac{S^i}{Ste}\left(\frac{\mathrm{d}S}{\mathrm{d}\tau}\right) + \frac{\lambda_l}{\lambda_s}\left(i - \frac{3S^i}{S-\delta}\right)\right] \\ & + (1-S)\left(5+S^i\right)\left[\frac{S^i}{6\cdot Ste}\left(\frac{\mathrm{d}S}{\mathrm{d}\tau}\right)^2 \right. \\ & \left. + \frac{\lambda_l}{\lambda_s}\frac{\alpha_l}{\alpha_s}\frac{S^i}{(S-\delta)^2}\right]\end{aligned} \tag{45}$$

Comparing the above equations for the two-region problem with corresponding equations for the one-region problem, they are of the same form except the heat layer thickness $\delta(\tau)$ term involved in Eqs. (42)–(45). Once $\delta(\tau)$ is solved, as below, the temperature and its gradient at the fixed boundary could be completely determined. Note that Eqs. (43)–(45) for the two-region are reduced to Eqs. (12)–(14) for the one-region problem, respectively, if $\lambda_l/\lambda_s = 0$.

Substituting Eqs. (38b) and (37) into the heat-balance integral equation (35) for liquid phase, we obtain the following first-order ordinary differential equation for the determination of $\delta(\tau)$:

$$\mu\frac{\mathrm{d}\mu}{\mathrm{d}S} - \left(4 - i\frac{\mu}{S}\right)\mu - 12\frac{\alpha_l}{\alpha_s}\left(\frac{\mathrm{d}S}{\mathrm{d}\tau}\right)^{-1} = 0 \tag{46}$$

where $\mu = S - \delta$. The solution of this equation is discussed in the next section.

2.2.3. The relation of $S(\tau)$ and $\delta(\tau)$

By solving Eq. (46) subject to the appropriate initial condition the relation between $S(\tau)$ and $\delta(\tau)$ could be determined. This equation is a nonlinear first-order ordinary differential equation. There is no general exact solution. We choose two cases in the following analysis, i.e., Cartesian geometry, and the phase change interface varies with time following the power law for $n = 1$ and 2, respectively.

(1) $i = 0$, $n = 1$, $S(\tau) = 1 - \beta\tau$: This case represents the phase change interface moving at a constant rate. Eq. (46) becomes

$$\mu\frac{\mathrm{d}\mu}{\mathrm{d}S} - 4\mu + \frac{12}{\beta}\frac{\alpha_l}{\alpha_s} = 0 \tag{47a}$$

with

$$\mu = 0 \quad \text{for } S = 1 \tag{47b}$$

The exact solution of Eq. (47) is

$$S = 1 + \frac{S-\delta}{4} + \frac{3\alpha_l}{4\beta\alpha_s}\ln\left(1 - \frac{\beta\alpha_s(S-\delta)}{3\alpha_l}\right) \tag{48}$$

This equation cannot yet be used to substitute into Eqs. (43)–(45) to calculate Θ_s, u_s, and $\partial u_s/\partial R$ directly, because $\delta(\tau)$ is implicit in the above equation. A numerical integral is needed to solve Θ_s from Eq. (43). The temperature and its gradient at the fixed boundary for this case can then be evaluated from Eqs. (44) and (45), respectively.

(2) $i = 0$, $n = 2$, $S(\tau) = 1 - (\beta\tau)^{1/2}$: This case is equivalent to the direct Stefan problem with the constant temperature at the fixed boundary. Eq. (46) is rewritten as

$$\mu\frac{\mathrm{d}\mu}{\mathrm{d}S} - 4\mu + \frac{24}{\beta}\frac{\alpha_l}{\alpha_s}(1-S) = 0 \tag{49a}$$

with

$$\mu = 0 \quad \text{for } S = 1 \tag{49b}$$

The exact solution of Eq. (49a) subject to condition (49b) results in the following relation between $S(\tau)$ and $\delta(\tau)$

$$S - \delta = \xi(1 - S) \tag{50a}$$

where

$$\xi = -2\left(1 - \sqrt{1 + \frac{6\alpha_l}{\beta\alpha_s}}\right) \tag{50b}$$

Substituting Eq. (50) into the ordinary differential equation (43) subject to the condition $\Theta_s = 0$ for $S = 1$, Θ_s is solved

$$\Theta_s = -\frac{\beta(1-S)}{\beta - 24}\left(\frac{6}{Ste} + \frac{36\lambda_l}{\xi\beta\lambda_s} + \frac{\beta}{2\cdot Ste} + \frac{12\lambda_l\alpha_l}{\xi^2\beta\lambda_s\alpha_s}\right) \tag{51}$$

Thus the temperature and its gradient at the fixed boundary can be solved from Eqs. (44) and (45), respectively

$$u_s(1,\tau) = -\frac{4\Theta_s}{1-S} + \frac{\beta}{2\cdot Ste} + \frac{3\lambda_l}{\xi\lambda_s} + \frac{\beta^2}{24\cdot Ste} + \frac{\lambda_l\alpha_l}{\xi^2\lambda_s\alpha_s} \tag{52}$$

$$\frac{\partial u_s(1,\tau)}{\partial R} = -\frac{12\Theta_s}{(1-S)^2} + \frac{1}{(1-S)} \times\left(\frac{5\beta}{2\cdot Ste} + \frac{15\lambda_l}{\xi\lambda_s} + \frac{\beta^2}{4\cdot Ste} + \frac{6\lambda_l\alpha_l}{\xi^2\lambda_s\alpha_s}\right) \tag{53}$$

If $\lambda_l/\lambda_s = 0$, Eqs. (52) and (53) for the two-region problem are reduced to Eqs. (23) and (24) for the one-region problem, respectively. Similar to the one-region problem the temperature at the fixed boundary is also a constant. No exact solution is available for this two-region inverse Stefan problem, namely inward solidification of a slab of finite thickness. An analytical solution of the corresponding direct Stefan problem obtained by a combination of the exact and the approximate integral method has been reported by Cho and Sunderland [19]. The expressions of the temperature and its gradient at the fixed boundary can be rewritten with present terms, respectively

$$u_s = -e^{\beta/4}erf\left(\frac{\sqrt{\beta}}{2}\right)\left(\frac{\sqrt{\pi\beta}}{2\cdot Ste} + \frac{\lambda_l}{Z_3\lambda_s}\sqrt{\frac{\alpha_s}{\alpha_l}}\right) \tag{54}$$

$$\frac{\partial u_s}{\partial R} = -\frac{e^{\beta/4}}{\sqrt{\pi\tau}}\left(\frac{\sqrt{\pi\beta}}{2\cdot Ste} + \frac{\lambda_l}{Z_3\lambda_s}\sqrt{\frac{\alpha_s}{\alpha_l}}\right) \tag{55}$$

where

$$Z_3 = -\frac{2}{3}\sqrt{\frac{\beta\alpha_s}{\pi\alpha_l}}\left(1 - \sqrt{1 + \frac{6\alpha_l}{\beta\alpha_s}}\right) \tag{56}$$

$erf(x)$ is an error function. Numerical results tabulated in Tables 4–6 show comparisons of both methods for different values of *Ste* and β. The relative deviation of the temperature is less than 2.3%; the temperature gradient is less than 1% within the range of *Ste* and β listed in the tables.

3. Summary and conclusions

The heat-balance integral method cannot only be applied to the Stefan problem, but can also be applied to the inverse Stefan problem effectively. In this paper, the heat-balance integral method has been employed to analyze the one-dimensional inverse Stefan problem in Cartesian and spherical geometry when the movement of the phase change interface is specified. The approximate analytic solutions expressed in finite forms could avoid the difficulties of estimating the convergence of the series solution. It is not required that the Stefan number, *Ste*, must be a small parameter as in the perturbation solution. The present method is both simple and accurate for predicting the necessary temperature and heat flux variation that would be required at the fixed boundary when the movement of the phase change interface is prescribed.

Table 4
The temperature at the fixed boundary for different values of *Ste* and β and $\lambda_l/\lambda_s = 1$, $\alpha_s/\alpha_l = 1$ (two-region problem, Cartesian geometry, $n = 2$)

β	$Ste = 0.05$		$Ste = 0.1$		$Ste = 0.5$		$Ste = 1$		$Ste = 5$	
	Cho [19]	present	Cho[19]	present	Cho[19]	present	Cho[19]	present	Cho[19]	present
0.04	−0.5365	−0.5380	−0.3351	−0.3366	−0.1740	−0.1756	−0.1539	−0.1554	−0.1378	−0.1393
0.1	−1.2408	−1.2444	−0.7324	−0.7359	−0.3256	−0.3292	−0.2748	−0.2784	−0.2341	−0.2377
0.5	−6.0644	−6.0685	−3.3452	−3.3536	−1.1700	−1.1816	−0.8981	−0.9101	−0.6805	−0.6929
1	−12.9256	−12.8664	−7.0026	−6.9787	−2.2643	−2.2686	−1.6720	−1.6798	−1.1981	−1.2088
2	−30.3298	−29.6439	−16.2229	−15.8561	−4.9374	−4.8258	−3.5267	−3.4470	−2.3982	−2.3439

Table 5
The temperature gradient at the fixed boundary for different β and $Ste = 1$, $\lambda_{ll}/\lambda_s = 1$, $\alpha_s/\alpha_l = 1$ (two-region problem, Cartesian geometry, $n = 2$)

$Ste = 1, \beta = 0.1$			$Ste = 1, \beta = 0.5$			$Ste = 1, \beta = 1$			$Ste = 1, \beta = 2$		
τ	Cho [19]	present	τ	Cho[19]	present	τ	Cho[19]	present	τ	Cho[19]	present
1	−0.87626	−0.87654	0.2	−2.9587	−2.9635	0.1	−5.7311	−5.7382	0.05	−13.0343	−12.9366
2	−0.61961	−0.61981	0.4	−2.0921	−2.0955	0.2	−4.0525	−4.0575	0.10	−9.2166	−9.1476
3	−0.50591	−0.50607	0.6	−1.7082	−1.7110	0.3	−3.3088	−3.3130	0.15	−7.5253	−7.4689
4	−0.43813	−0.43827	0.8	−1.4794	−1.4817	0.4	−2.8655	−2.8691	0.20	−6.5171	−6.4683
5	−0.39188	−0.39200	1.0	−1.3232	−1.3253	0.5	−2.5630	−2.5662	0.25	−5.8291	−5.7854
6	−0.35773	−0.35785	1.2	−1.2079	−1.2098	0.6	−2.3397	−2.3426	0.30	−5.3212	−5.2813
7	−0.33120	−0.33130	1.4	−1.1183	−1.1201	0.7	−2.1661	−2.1688	0.35	−4.9265	−4.8896
8	−0.30980	−0.30990	1.6	−1.0461	−1.0477	0.8	−2.0262	−2.0288	0.40	−4.6083	−4.5738
9	−0.29209	−0.29218	1.8	−0.98624	−0.98782	0.9	−1.9104	−1.9127	0.45	−4.3448	−4.3122
10	−0.27710	−0.27719	2.0	−0.93563	−0.93713	1.0	−1.8123	−1.8146	0.50	−4.1218	−4.0909

Table 6
The temperature gradient at the fixed boundary for different *Ste* and $\beta = 1$, $\lambda_l/\lambda_s = 1$, $\alpha_s/\alpha_l = 1$ (two-region problem, Cartesian geometry, $n = 2$)

$Ste = 0.05, \beta = 1$			$Ste = 0.1, \beta = 1$			$Ste = 0.5, \beta = 1$			$Ste = 5, \beta = 1$		
τ	Cho [19]	present	τ	Cho[19]	present	τ	Cho[19]	present	τ	Cho[19]	present
0.1	−44.3053	−44.2699	0.1	−24.0031	−23.9901	0.1	−7.7613	−7.7662	0.1	−4.1069	−4.1158
0.2	−31.3286	−31.3035	0.2	−16.9727	−16.9635	0.2	−5.4881	−5.4915	0.2	−2.9040	−2.9103
0.3	−25.5797	−25.5592	0.3	−13.8582	−13.8507	0.3	−4.4810	−4.4838	0.3	−2.3711	−2.3763
0.4	−22.1526	−22.1349	0.4	−12.0015	−11.9950	0.4	−3.8806	−3.8831	0.4	−2.0534	−2.0579
0.5	−19.8139	−19.7981	0.5	−10.7345	−10.7287	0.5	−3.4710	−3.4732	0.5	−1.8367	−1.8407
0.6	−18.0876	−18.0731	0.6	−9.7992	−9.7939	0.6	−3.1685	−3.1705	0.6	−1.6766	−1.6803
0.7	−16.7458	−16.7324	0.7	−9.0723	−9.0674	0.7	−2.9335	−2.9354	0.7	−1.5523	−1.5556
0.8	−15.6643	−15.6518	0.8	−8.4864	−8.4818	0.8	−2.7440	−2.7458	0.8	−1.4520	−1.4552
0.9	−14.7684	−14.7566	0.9	−8.0010	−7.9967	0.9	−2.5871	−2.5887	0.9	−1.3690	−1.3719
1.0	−14.0106	−13.9994	1.0	−7.5904	−7.5863	1.0	−2.4543	−2.4559	1.0	−1.2987	−1.3015

Several numerical illustrations for one-region and two-region problems, wherein the position of the phase change interface varies with time following a power law, namely $S(\tau) = 1 - (\beta\tau)^{1/n}$, are given in the paper. The results are in good agreement compared with the exact and/or numerical solutions given in the literature. For the plane inverse Stefan problem ($i = 0$), it is shown that the maximum error between the exact and the heat-balance integral approximate solution is not larger than 2.3% when $\beta \leqslant 2$. The temperature at the fixed boundary, $u(1,\tau)$, maintains a constant throughout for $n = 2$, decreases with time for $n < 2$ and increases monotonically with time from $-\infty$ when $n > 2$. For the spherical inverse Stefan problem with $n = 1$, $u(1,\tau)$ will first decrease and then increase with time from the phase change temperature when the phase change interface is moving at a constant rate all along. As a result, $u(1,\tau)$ is returned to the phase change temperature even though the entire sphere has not yet been solidified completely.

In fact, the heat-balance integral method is a specific case of the weighted residual method (i.e., the weighting function is taken to be unit) [10], which could be applied to the inverse Stefan problem in order to improve the accuracy of the solution when β is larger. Using the concept of more thermal layers may be of aid as illustrated in the application to the direct phase change problem [12,14]. Further investigation could extend the application of the heat-balance integral into the case of a binary system, there may exist a two-phase region between the purely solid and purely liquid phase. The inverse problem of this type is a three-region problem.

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